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LETTER TO THE EDITOR

Directed bond percolation on the honeycomb lattice

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Abstract. Using a transfer matrix method we derive a series expansion for the percolation probability on the directed honeycomb lattice. The high-density series is obtained to order q^{13}. A Padé approximant analysis of the series has been used to estimate the percolation threshold q_c and the critical exponent β.

Since the percolation process was introduced by Broadbent and Hammersley (1957) and Domb (1959) it has been studied very intensely (Stauffer 1985). Percolation now forms an important branch of critical phenomena theory.

Lattices in which directed bonds are independently present with probability p and absent with probability $q = 1 - p$ we shall refer to as directed lattices. Directed percolation can be associated with a great number of physical problems: Reggeon field theory (Grassberger and Sundermeyer 1978, Grassberger and de la Torre 1979, Cardy and Sugar 1980), three-dimensional random resistor-diode networks (Redner and Brown 1981) and galactic evolution (Schulman and Seiden 1982). It can also be interpreted as a model for spreading under some influence or biased direction like epidemic models or a forest fire and it does not belong to the same universality class as the isotropic (undirected) case (Blease 1977a, b).

As translational invariance is completely destroyed in the directed version the theory of conformal invariance cannot be applied (Essam *et al* 1988) and it may be possible that their critical exponents are not simple rational fractions. Besides, some recent works have given very accurate estimates of the critical probabilities and a corresponding improvement in the conjectured values of the exponents (Essam *et al* 1988, Baxter and Guttmann 1988, Grassberger 1989).

Series expansions for the moments of the pair connectedness (low-density) have now been performed on most of the usual lattices (Blease 1977b, Essam *et al* 1988). However, in two dimensions, the corresponding series expansions for the percolation probability (high-density) are only available for the square and triangular lattices. In particular, for the honeycomb lattice the perimeter method cannot be applied (Blease 1977a) and series expansions for the percolation probability has remained unknown for this lattice. Following Baxter and Guttmann (1988) we use a transfer matrix method which allows us to determine this series to order q^{13}. Below we present the method in a succinct form.

Consider a honeycomb lattice drawn as in figure 1. Two sites are connected if one can walk along bonds linking these sites always in the allowed directions. For q less than a critical value q_c and for an infinite system there is a non-zero probability $P(q)$

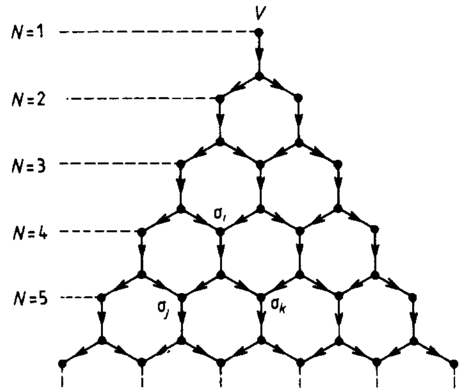

Figure 1. The directed honeycomb lattice showing the variables σ_i, σ_j, σ_k whose interaction weight function is $W(\sigma_i, \sigma_j, \sigma_k)$. The rows are also indicated.

that a given site V is connected to an infinite cluster. Now let $P_N(q)$ be the probability that the apex V is connected to at least one site in the row N. Then we expect that

$$P(q) = \lim_{N\to\infty} P_N(q). \tag{1}$$

Let us associate with each site j an Ising variable σ_j such that $\sigma_j = +1$ if j is connected to at least one site in the row N and $\sigma_j = -1$ otherwise. If we define a function $W(\sigma_i, \sigma_j, \sigma_k)$ as being the probability that site i is in state σ_i, given that sites j, k are in states σ_j, σ_k (see figure 1) and a function $f(\sigma_1)$ as corresponding to the probability that the apex V is in state σ_1 and finally, if we assign the value $+1$ to all sites in the last row then it follows that

$$P_N(q) = f(+1) \tag{2}$$

$$f(\sigma_1) = \sum_{\{\sigma\}} \prod_j W(\sigma_i, \sigma_j, \sigma_k) \tag{3}$$

where the product is over all sites j that are above the bottom row and the sum is over all possible values ± 1 excluded the topmost spin σ_1:

$$W(\sigma_i, \sigma_j, \sigma_k) = \tfrac{1}{2}(1-\sigma_i) + \tfrac{1}{4}\{\sigma_i(1-q)^2[(q-1)\sigma_j\sigma_k + (q+1)(\sigma_j+\sigma_k) + q + 3]\} \tag{4}$$

and (for a finite lattice of N rows)

$$P_N(q) = \sum_{m=0}^{3N(N-1)/2} \alpha_N m q^m. \tag{5}$$

It is easy to show that

$$P_1(q) = 1 \qquad P_2(q) = 1 - q - q^2 + q^3$$

$$P_3(q) = 1 - q - 4q^2 + 4q^3 + 8q^4 - 12q^5 + 8q^7 - 5q^8 + q^9$$

$$\begin{aligned} P_4(q) = 1 &- q - 4q^2 - 12q^3 + 63q^4 - 23q^5 - 192q^6 + 284q^7 + 40q^8 - 421q^9 + 317q^{10} \\ &+ 112q^{11} - 305q^{12} + 151q^{13} + 38q^{14} - 80q^{15} + 41q^{16} - 10q^{17} + q^{18}. \end{aligned} \tag{6}$$

For large N we wrote a REDUCE (Hearn 1987) program and we were able to obtain $P_{14}(q)$. We found ourselves in a situation which resembles that of the directed square lattice (Baxter and Guttmann 1988): going from N to $N+1$ leaves the coefficient

Table 1. Dlog Padé approximants to the percolation probability series for directed bond percolation on the honeycomb lattice. The entries give q_c (left) and β (right) estimates.

N	$[(N-1)/N]$		$[N/N]$		$[(N+1)/N]$	
3	0.176 48	0.267	0.176 75	0.270	0.176 80	0.270
4	0.176 85	0.271	0.177 16	0.276	0.177 01	0.273
5	0.177 02	0.273	0.177 01	0.274	0.177 11	0.275
6	0.177 11	0.275	0.177 09	0.274		

Table 2. Ratio method applied to the percolation probability series. Entries to the left (right) are $q_c(\beta)$ estimates.

N	$(N, N-1)$		$(N, N-2)$	
7	0.177 99	0.256	0.175 77	0.328
8	0.177 04	0.287	0.177 51	0.269
9	0.177 56	0.267	0.177 30	0.278
10	0.176 78	0.301	0.177 17	0.282
11	0.177 38	0.272	0.177 08	0.288
12	0.176 90	0.298	0.177 14	0.283
13	0.177 02	0.290	0.176 96	0.294

Table 3. Padé approximants to the series generates by $[P(q)]^{1/\beta}$ giving the q_c estimates.

N	$[N/(N-1)]$	$[N/N]$	$[N/(N+1)]$
4	0.177 23	0.177 21	0.177 19
5	0.177 19	0.177 11	0.177 15
6	0.177 15	0.177 16	0.177 15
7	0.177 15		

of $1, q, \ldots, q^{N-1}$ (q^N for the square lattice) unchanged so that the percolation probability can be written

$$P(q) = P_\infty(q) = \sum_{N=1}^{\infty} a_{NN-1} q^{N-1} \tag{7}$$

and we obtain

$$P(q) = 1 - q - 4q^2 - 12q^3 - 45q^4 - 188q^5 - 835q^6 - 3849q^7 - 18\,242q^8 - 88\,265q^9 - 434\,295q^{10} - 2165\,198q^{11} - 10\,915\,089q^{12} - 55\,534\,781q^{13} \ldots \tag{8}$$

For the square lattice Baxter and Guttmann have found a remarkable property involving some linear combinations of the Catalan numbers and the coefficients of the series expansion. They used this fact in order to extrapolate the series. Regrettably, we were unable to find any similar situation for the honeycomb lattice.

We have used Dlog Padé approximants and the ratio method to obtain estimates of the critical probability q_c and exponent β. The results of our analysis are shown in tables 1 and 2 and they favour the Padé method which exhibits faster convergence

(Dlog Padé approximants are usually well suited to study order parameter series). From these we have the estimates: $\beta = 0.273 \pm 0.006$ and $q_c = 0.1770 \pm 0.0005$. Although completely consistent with universality, our values for the exponent β are poor when compared with earlier estimates (Baxter and Guttmann 1988, $\beta = 0.2764 \pm 0.0001$) obtained from longer series expansions on the square lattice.

If we accept the value $\beta = 0.2764$ then the estimate q_c can be improved by writing Padé approximants to the series $[P(q)]^{1/\beta}$ which now has a simple zero at $q = q_c$. The results are presented in table 3. Taking the confidence limits as the apparent scatter of q_c we conclude that $q_c = 0.177\,17 \pm 0.000\,06$ which is more precise than previous estimates (Blease 1977b) by one order of magnitude.

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